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# Finite-Dimensional Relativistic Quantum Mechanics

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A finite-dimensional relativistic quantum mechanics is developed by first quantizing Minkowski space. Two-dimensional space-time event observables are defined and quantum microscopic causality is studied. Three-dimensional colored even observables are introduced and second quantized on a representation space of the restricted Poincaré group. Creation, annihilation, and field operators are introduced and a finite-dimensional Dirac theory is presented.

## 1. INTRODUCTION

Although finite-dimensional nonrelativistic quantum mechanics has been studied by various authors [2–7], this work appears to be the first to consider the relativistic situation for a finite-dimensional theory. We begin by quantizing Minkowski four-space in terms of a two-dimensional complex space $\mathbb{C}^2$. This is not a new technique [8]. What seems to be new, however, is our interpretation of the self-adjoint matrices on $\mathbb{C}^2$ as space-time event observables and our study of quantum microscopic causality. A relativistic quantum theory, in one widespread usage, is one with an action of the Poincaré group as automorphisms of the algebra of observables. This is not so; the observables are mapped as forms are, rather than as operators, and their products are not preserved.

We next introduce colored event observables on $\mathbb{C}^3$ and the second quantization of $\mathbb{C}^3$ on a representation space of the restricted Poincaré group. This results in a finite-dimensional relativistic quantum field theory. Creation, annihilation, and field operators are discussed. Finally, we present a finite-dimensional Dirac theory. Possible applications to a quark model are mentioned.

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## 2. EVENT OBSERVABLES

Let $M^4$ be the Minkowski space consisting of 4-tuples $(x_0, x_1, x_2, x_3) \in \mathbb{R}$ with the indefinite form

$$x \cdot y = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3$$

We use the notation $x \cdot y = x_0 y_0 - \mathbf{x} \cdot \mathbf{y}$, where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. Let $L$ be the restricted Lorentz group on $M^4$. That is, $L$ is the group of real linear transformations $\Lambda$ on $M^4$ satisfying $\Lambda x \cdot \Lambda y = x \cdot y$ for every $x, y \in M^4$, $\det \Lambda = 1$, and $\Lambda_{00} \geq 0$. The *restricted Poincaré group* $\mathcal{P}$ is the set $\{(a, \Lambda): a \in M^4, \Lambda \in L\}$ with group operation

$$(a_1, \Lambda)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2)$$

Let $\mathbb{C}^2 = \{(\phi_1, \phi_2): \phi_1, \phi_2 \in \mathbb{C}\}$ with the standard inner product $\langle \phi, \psi \rangle = \phi_1 \bar{\psi}_1 + \phi_2 \bar{\psi}_2$, $\phi = (\phi_1, \phi_2)$, $\psi = (\psi_1, \psi_2) \in \mathbb{C}^2$. We regard $\mathbb{C}^2$ as a two-dimensional quantum mechanical Hilbert space whose unit vectors correspond to pure states and whose set of self-adjoint matrices $\Omega$ correspond to observables. As we shall see, $\mathbb{C}^2$ gives a quantization of space-time. The special linear group $SL(2, \mathbb{C})$ is called the *transformation group* of $\mathbb{C}^2$. For $A \in SL(2, \mathbb{C})$ and $Q \in \Omega$, define $T(A)Q = AQA^*$. It is easy to check that $T$ is a group homomorphism from $SL(2, \mathbb{C})$ into the group of determinant-preserving invertible linear operators on the real linear space $\Omega$. Define the set

$$\mathcal{P}(2, \mathbb{C}) = \{(P, A): P \in \Omega, A \in SL(2, \mathbb{C})\}$$

Then $\mathcal{P}(2, C)$ becomes a group (the *inhomogeneous transformation group*) under the operation

$$(P_1, A_1)(P_2, A_2) = (P_1 + T(A_1)P_2, A_1 A_2)$$

For $(P, A) \in \mathcal{P}(2, \mathbb{C})$ and $Q \in \Omega$, define $\tilde{T}(P, A)Q = P + T(A)Q$. Then

$$\begin{aligned} \tilde{T}(P_1, A_1)\tilde{T}(P_2, A_2)Q &= \tilde{T}(P_1, A_1)(P_2 + T(A_2)Q) \\ &= P_1 + T(A_1)P_2 + T(A_1 A_2)Q \\ &= \tilde{T}(P_1 + T(A_1)P_2, A_1 A_2)Q \\ &= \tilde{T}[(P_1, A_1)(P_2, A_2)]Q \end{aligned}$$

Hence, $\tilde{T}$ is a group homomorphism from $P(2, \mathbb{C})$ into the group of invertible transformations on $\Omega$. We interpret $\tilde{T}(P, A)Q$ as the observable $Q$ after the transformation corresponding to $(P, A)$ is made.

Define $\tau_j \in \Omega$, $j = 0, 1, 2, 3$, by

$$\tau_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

We call $\tau_0$ the *time observable* and $\tau_j$, $j = 1, 2, 3$, the $j$th *space observable.* For $x \in M^4$, define $\hat{x} \in \Omega$ by

$$\hat{x} = \sum_{j=0}^{3} x_j \tau_j$$

Then $\hat{}$ is a real linear bijection from $M^4$ to $\Omega$ which satisfies $\det \hat{x} = x \cdot x \equiv x^2$ for all $x \in M^4$. We denote the inverse of $\hat{}$ by $\check{}$ and note that $\check{Q}_j = \frac{1}{2}\operatorname{Tr}(Q, \tau_j)$, $j = 0, 1, 2, 3$. We call $\hat{x}$ an *event observable* and interpret $\hat{x}$ as the observable which measures the event $x \in M^4$.

For $A \in SL(2, \mathbb{C})$, define $\Lambda(A) \in L$ by $\Lambda(A)x = (A\hat{x}A^*)^\vee = [T(A)\hat{x}]^\vee$. The map $\Lambda: SL(2, \mathbb{C}) \to L$ is a group homomorphism onto $L$ and in fact $SL(2, \mathbb{C})$ is the universal covering group of $L$[8]. We extend $\Lambda$ to $\mathscr{P}(2, \mathbb{C})$ by defining $\tilde{\Lambda}: \mathscr{P}(2, \mathbb{C}) \to \mathscr{P}$, where $\tilde{\Lambda}(P, A)x = [\tilde{T}(P, A)\hat{x}]^\vee$. Notice that

$$\tilde{\Lambda}(P, A)x = [P + T(A)\hat{x}]^\vee = \check{P} + \Lambda(A)x = (\check{P}, \Lambda(A))x$$

The following shows that the surjection $\Lambda$: $\mathscr{P}(2, \mathbb{C}) \to \mathscr{P}$ is a group homomorphism:

$$\begin{aligned}\tilde{\Lambda}(P_1, A_1)\tilde{\Lambda}(P_2, A_2) &= (\check{P}_1, \Lambda(A_1))(\check{P}_2, \Lambda(A_2)) \\ &= [\check{P}_1 + \Lambda(A_1)\check{P}_2, \Lambda(A_1A_2)] \\ &= [(P_1 + T(A_1)P_2)^\vee, \Lambda(A_1A_2)] \\ &= \tilde{\Lambda}(P_1 + T(A_1)P_2, A_1A_2) \\ &= \tilde{\Lambda}[(P_1, A_1)(P_2, A_2)]\end{aligned}$$

We interpret $x \to \hat{x}$ as a two-dimensional quantum field and the relation $[(\check{P}, \Lambda(A))x]^\wedge = \tilde{T}(P, A)\hat{x}$ can be thought of as relativistic covariance.

*Theorem 1.* (a) The eigenvalues of $\hat{x}$ are $\lambda_\pm = x_0 \pm \mathbf{x} \cdot \mathbf{x}^{1/2}$. The corresponding eigenvectors are $\phi_\pm = (1, (\pm\mathbf{x} \cdot \mathbf{x}^{1/2} - x_3)/(x_1 + ix_2))$ unless $x_1 = x_2 = 0$ in which case $\phi_+ = (1, 0)$, $\phi_- = (0, 1)$.

(b) $x$ is timelike if and only if $\hat{x}$ is positive or negative, $x$ is spacelike if and only if $\hat{x}$ is neither positive or negative, $x$ is lightlike if and only if $\hat{x}$ has eigenvalue 0.

(c) $\hat{x}\hat{y} = \hat{y}\hat{x}$ if and only if $\{\mathbf{x}, \mathbf{y}\}$ is linearly dependent.

*Proof.* The proof of this is straightforward. ■

*Corollary 2.* If $x_0 \geq 0$, then $x$ is timelike if and only if $\hat{x}$ is positive. If $x_0 \leq 0$, then $x$ is timelike if and only if $\hat{x}$ is negative.

For $x \in M^4$ we call the set $\{y \in M^4$: $\mathbf{y} = \lambda\mathbf{x}$, $\lambda \in \mathbb{R}\}$ a *light plane.* We call the condition that $[\hat{x}, \hat{y}] = 0$ if and only if $x$ and $y$ lie on a common light plane *quantum microscopic causality.* It is our contention that quantum microscopic causality should be used instead of microscopic causality which states that quantum fields with spacelike separated supports commute.

We now describe when $\hat{x}$ is a projection. As far as the trivial projections are concerned, $\hat{x}=0$ if and only if $x=0$ and $\hat{x}=I$ if and only if $x=(1,0,0,0)$. The other projections on $\mathbb{C}^2$ are one-dimensional and correspond to pure states in $\mathbb{C}^2$.

*Theorem 3.* (a) $\hat{x}$ is a nontrivial projection if and only if $x$ is lightlike and $x_0=\frac{1}{2}$.

(b) If $\hat{x}$ and $\hat{y}$ are nontrivial projections then $\hat{x}\hat{y}=0$ if and only if $\mathbf{x}=-\mathbf{y}$.

*Proof.* (a) If $x$ is lightlike and $x_0=\frac{1}{2}$, then $x=(\frac{1}{2},\mathbf{x})$, where $\mathbf{x}^2=\frac{1}{4}$. Hence,

$$\hat{x}^2=\begin{bmatrix} {x_0}^2+2x_0x_3+\mathbf{x}^2 & 2(x_1-ix_2)x_0 \\ 2(x_1+ix_2)x_0 & {x_0}^2-2x_0x_3+\mathbf{x}^2 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{2}+x_3 & x_1-ix_2 \\ x_1+ix_2 & \frac{1}{2}-x_3 \end{bmatrix}=\hat{x}$$

and we conclude that $\hat{x}$ is a nontrivial projection. Conversely, suppose $\hat{x}$ is a nontrivial projection. Then $\hat{x}^2=\hat{x}$ and at least one of the three numbers $x_1, x_2, x_3$ must be nonzero. If $x_3\neq 0$, then

$$x_0+x_3={x_0}^2+2x_0x_3+\mathbf{x}^2$$

$$x_0-x_3={x_0}^2-2x_0x_3+\mathbf{x}^2$$

imply that $2x_3=4x_0x_3$ and $x_0={x_0}^2+\mathbf{x}^2$. Hence, $x=\frac{1}{2}$ and $\mathbf{x}^2=\frac{1}{4}$. If $x_1$ or $x_2$ is nonzero, then

$$x_1-ix_2=2(x_1-ix_2)x_0$$

implies that $x_0=\frac{1}{2}$ and from the above, $\mathbf{x}^2=\frac{1}{4}$.

(b) Suppose that $\hat{x}$ and $\hat{y}$ are nontrivial projections and $\mathbf{x}=-\mathbf{y}$. Applying Theorem 1, $\hat{x}$ and $\hat{y}$ must commute. But two nonequal one-dimensional projections commute if and only if they are orthogonal. Conversely, suppose that $\hat{x}$ and $\hat{y}$ are nontrivial projections and $\hat{x}\hat{y}=0$. Since $\hat{x}$ and $\hat{y}$ commute, it follows from Theorem 1 that $\mathbf{y}=\lambda\mathbf{x}$ for some $\lambda\in\mathbb{R}$. Since $\hat{x}\hat{y}=0$, we have

$$\lambda(x_0+x_3)(x_1-ix_2)+(x_1-ix_2)(x_0-\lambda x_3)=0$$

If $x_1$ or $x_2$ is nonzero, then $\lambda x_0+x_0=0$, so $\lambda=-1$. Now suppose that $x_1=x_2=0$ and hence $x_3=\pm\frac{1}{2}$. Then $\hat{x}\hat{y}=0$ implies that

$$(x_0+x_3)(x_0+\lambda x_3)=(x_0-x_3)(x_0-\lambda x_3)=0$$

If $x_3=\frac{1}{2}$, the first term gives $\lambda=-1$ and if $x_3=-\frac{1}{2}$, the second term gives $\lambda=-1$. ■

We have seen that the pure states in $\mathbb{C}^2$ corresponds to lightlike events $x$ for which $x_0=\frac{1}{2}$. Since mixed states are convex combinations of pure states, it follows that mixed states in $\mathbb{C}^2$ correspond to timelike events $x$ for which $x_0 = \frac{1}{2}$. Another simple proof of this result is now given.

*Corollary 4.* $\hat{x}$ is a density matrix if and only if $x$ is timelike and $x_0 = \frac{1}{2}$.

*Proof.* By definition, $\hat{x}$ is a density matrix and only its eigenvalues $\lambda_+$, $\lambda_-$ are nonnegative and sum to 1. Applying Theorem 1, this is equivalent to $x$ being timelike and

$$1 = \lambda_+ + \lambda_- = x_0 + \mathbf{x}\cdot\mathbf{x}^{1/2} + x_0 - \mathbf{x}\cdot\mathbf{x}^{1/2} = 2x_0 \quad \blacksquare$$

We call $x \in M^4$ a *simple event* if $\hat{x}$ is a nontrivial projection. Two simple events $x$, $y$ are called *orthogonal* if $\hat{x}\hat{y} = 0$.

*Corollary 5.* (a) If $x$, $y$ are simple events, the following statements are equivalent: (1) $x$ and $y$ are orthogonal; (2) $\mathbf{x} = -\mathbf{y}$; (3) $x$ and $y$ lie on a common light plane.

(b) Every $x \in M^4$ for which $\mathbf{x}^2 \neq 0$ has a unique representation as a linear combination of two orthogonal simple events. In fact

$$x = (x_0 + \mathbf{x}\cdot\mathbf{x}^{1/2})(\tfrac{1}{2}, a_1, a_2, a_3) + (x_0 - \mathbf{x}\cdot\mathbf{x}^{1/2})(\tfrac{1}{2}, -a_1, -a_2, -a_3)$$

where $a_i = x_i/2\mathbf{x}\cdot\mathbf{x}^{1/2}$, $i = 1, 2, 3$.

Of course, if $\mathbf{x}^2 = 0$, then $x = x_0(1, 0, 0, 0)$.

## 3. COLORED OBSERVABLES

In this section we extend the results of Section 2 to $V = \mathbb{C}^3$. Although $V$ can describe any three-dimensional quantum system such as a spin-one system, we shall draw our analogy from the "color space" for a quark model [1, 2]. Let $c_1, c_2, c_3 \in \mathbb{R}$ be fixed distinct numbers corresponding to color values. These numbers were computed in [2] but their specific values are not important for our present study. Let $e_1, e_2, e_3$ be the standard basis for $V$. A self-adjoint matrix on $\mathbb{C}^3$ with eigenvalue $c_1$ and corresponding eigenvector $e_1$ is called a *red observable*. Denote the set of red observables by $\Omega_r$ and define the set of yellow observables $\Omega_y$ and blue observables $\Omega_b$ in an analogous way. The *color observable* $C = \mathrm{diag}(c_1, c_2, c_3)$ is the unique observable in $\Omega_r \cap \Omega_y \cap \Omega_b$. For $x \in M^4$ we define the *red*, *yellow*, and *blue* event observables $x_r \in \Omega_r$, $x_y \in \Omega_y$, $x_b \in \Omega_b$, respectively, as follows:

$$x_r = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & x_0 + x_3 & x_1 - ix_2 \\ 0 & x_1 + ix_2 & x_0 - x_3 \end{bmatrix}, \qquad x_y = \begin{bmatrix} x_0 + x_3 & 0 & x_1 - ix_2 \\ 0 & c_2 & 0 \\ x_1 + ix_2 & 0 & x_0 + x_3 \end{bmatrix}$$

$$x_b = \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 & 0 \\ x_1 + ix_2 & x_0 - x_3 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

Results analogous to those in Theorem 1 hold. The eigenvalues of $x_r$ are $\lambda_0 = c_1$ and $\lambda_\pm = x_0 + \mathbf{x}\cdot\mathbf{x}^{1/2}$. The corresponding eigenvectors are $\phi_0 = (1, 0, 0)$, $\phi_\pm = (0, 1, (\pm\mathbf{x}\cdot\mathbf{x}^{1/2} - x_3)/(x_1 + ix_2))$ unless $x_1 = x_2 = 0$, in which case $\phi_+ = (0, 1, 0)$, $\phi_- = (0, 0, 1)$. The red observables $x_r$, $y_r$ commute if and only if $\{\mathbf{x}, \mathbf{y}\}$ is linearly dependent. Similar results holds for $x_y$ and $x_b$. Different color observables do not commute except under degenerate circumstances. For example, it can be shown that $x_b$ and $z_y$ commute if and only if $x_1 = x_2 = z_1 = z_2 = 0$ or $x_1 = x_2 = 0$ and $x_0 + x_3 = c_3$ or $z_1 = z_2 = 0$ and $z_0 + z_3 = c_2$.

If $A$ is a $2\times 2$ complex matrix and $\lambda \in \mathbb{R}$, we use the notation

$$A_{\lambda,r} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & & \\ & & A \\ 0 & & \end{bmatrix}$$

We also use the analogous notation $A_{\lambda,y}$ and $A_{\lambda,b}$. Thus, $x_r = \hat{x}_{c_1,r}$, $x_y = \hat{x}_{c_2,y}$, $x_b = \hat{x}_{c_3,b}$, $x \in M^4$. We define the groups

$$SL(2, \mathbb{C})_r = \{A_{1,r} \colon A \in SL(2, \mathbb{C})\}$$

$$\mathcal{P}(2, C)_r = \{(P_{0,r}, A_{1,r}) \colon P \in \Omega, A \in SL(2, \mathbb{C})\}$$

We also define $SL(2, \mathbb{C})_y$, $SL(2, \mathbb{C})_y$, and $\mathcal{P}(2, \mathbb{C})_b$ in analogous ways. For $A \in SL(2, \mathbb{C})_r$, $Q \in \Omega_r$, define $T(A)Q = AQA^*$ and for $(P, A) \in \mathcal{P}(2, \mathbb{C})_r$ define $\tilde{T}(P, A)Q = P + T(A)Q$. Also, define $\tilde{\Lambda}_r \colon \mathcal{P}(2, \mathbb{C})_r \to \mathcal{P}$ by $\tilde{\Lambda}(P_{0,r}, A_{1,r}) = (\check{P}, \Lambda(A))$ and analogously for $\tilde{\Lambda}_y$, $\tilde{\Lambda}_b$. Most of the results of Section 2 now hold.

## 4. FIELD THEORY

For $m \in \mathbb{R}$, let $H_m = \{p \in M^4 \colon p\cdot p = m^2, p_0 > 0\}$ be the *mass hyperboloid.* Then $H_m$ is invariant under $L$ [8]. Let $j_m$ be the homeomorphism of $H_m$ onto $\mathbb{R}^3$ (if $m = 0$, onto $R^3 - \{0\}$) given by $j_m(p) = \mathbf{P}$. Define the measure $\mu_m$ on $H_m$ by

$$\mu_m(E) = \int_{j_m(E)} (m^2 + \mathbf{p}^2)^{-1/2}\, d^3\mathbf{p}$$

for every measurable set $E \subseteq H_m$. It can be shown that $\mu_m$ is the unique $L$-invariant measure on $H_m$ up to a multiplicative constant [8]. Every irreducible unitary representation of $\mathcal{P}$ is equivalent to a representation $\mathcal{U}(a, \Lambda)$ on $L^2(H_m, d\mu_m)$ [8] where

$$[\mathcal{U}(a, \Lambda)f](p) = e^{ip\cdot a} f(\Lambda^{-1}p)$$

It follows that every irreducible unitary representation of $\mathcal{P}(2, \mathbb{C})$ is equivalent to a representation $\mathcal{U}(P, A)$ on $L^2(H_m, d\mu_m)$ where

$$[\mathcal{U}(P, A)f](p) = e^{i\check{P}\cdot p}f(\Lambda(A)^{-1}p)$$

For the group $\mathcal{P}(2, \mathbb{C})_r$ we have the above representation for $\mathcal{U}_r(P_{0,r}, A_{1,r}) \equiv \mathcal{U}(P, A)$ and similarly for $\mathcal{P}(2, \mathbb{C})_y$ and $\mathcal{P}(2, \mathbb{C})_b$. Furthermore, for the finite-dimensional quark model in [2], $m$ has only six possible values given by the six flavor values.

Let $h_j$, $j = 0, 1, \ldots,$ be the Hermite functions on $\mathbb{R}$. Then the functions $h_{j_1,j_2,j_3}(p_1, p_2, p_3) = h_{j_1}(p_1)h_{j_2}(p_2)h_{j_3}(p_3), j_1, j_2, j_3 = 0, 1, \ldots,$ form an orthonormal basis for $L^2(\mathbb{R}^3, d^3\mathbf{p})$. It follows that the functions $g_{j_1,j_2,j_3}(p) = (m^2 + \mathbf{p}^2)^{1/4}h_{j_1,j_2,j_3}(\mathbf{p})$, $j_1, j_2, j_3 = 0, 1, \ldots,$ form an orthonormal basis for $L^2(H_m, d\mu_m)$. We define the *annihilation* and *creation* operators $\Psi(1), \Psi^*(1)$, respectively, by

$$\Psi(1)g_{j_1,j_2,j_3} = k_1^{1/2}g_{j_1-1,j_2,j_3}$$
$$\Psi^*(q)g_{j_1,j_2,j_3} = (j_1+1)^{1/2}g_{j_1+1,j_2,j_3}$$

and extend by linearity to a dense subspace of $L^2(H_m, d\mu_m)$. Similar definitions hold for $\Psi(2)$, $\Psi(3)$, $\Psi^*(2)$, $\Psi^*(3)$. Then $\Psi(j)$ and $\Psi^*(j)$, $j = 1, 2, 3$, can be thought of as operator-valued, three-component vectors. We interpret $g_{j_1,j_2,j_3}$ as the state of $j_1 + j_2 + j_3$ Boson particles each of which has three possible states and the aggregate contains $j_1$ particles in the first state, $j_2$ in the second state, and $j_3$ in the third state. Then $\Psi(j)$ annihilates a particle in the $j$th state, while $\Psi^*(j)$ creates a particle in the $j$th state, $j = 1, 2, 3$.

This particle interpretation becomes evident by noting that $L^2(H_m, d\mu_m)$ is naturally isomorphic to the symmetric Fock space

$$SV = \mathbb{C} \oplus V \oplus (V \circledS V) \oplus (V \circledS V \circledS V) \oplus \cdots$$

where $V = \mathbb{C}^3$ and $\circledS$ denotes the symmetric tensor product. If $f_1, f_2, f_3$ is an orthonormal basis for $V$, a unitary isomorphism is given by

$$J_f(f_1^{j_1} \circledS f_2^{j_2} \circledS f_3^{j_3}) = g_{j_1,j_2,j_3}$$

The following lemma represents $\Psi(j)$ and $\Psi^*(j)$ as differential operators.

*Lemma 6.* On a dense subspace of $L^2(H_m, d\mu_m)$ we have for $j = 1, 2, 3$

$$\Psi(j) = 2^{-1/2}\left[p_j - \tfrac{1}{2}(\mathbf{p}^2 + m^2)^{-1}p_j + \frac{\partial}{\partial p_j}\right]$$
$$\Psi^*(j) = 2^{-1/2}\left[p_j + \tfrac{1}{2}(\mathbf{p}^2 + m^2)^{-1}p_j - \frac{\partial}{\partial p_j}\right]$$

*Proof.* It is well known that

$$2^{-1/2}\left(p_1+\frac{\partial}{\partial p_1}\right)h_{j_1,j_2,j_3}=j_1^{1/2}h_{j_1-1,j_2,j_3}$$

Hence,

$$\begin{aligned}2^{-1/2}\left(p_1+\frac{\partial}{\partial p_1}\right)g_{j_1,j_2,j_3}(p)&=2^{-1/2}\left(p_1+\frac{\partial}{\partial p_1}\right)(\mathbf{p}^2+m^2)^{1/4}h_{j_1,j_2,j_3}(\mathbf{p})\\&=(\mathbf{p}^2+m^2)^{1/4}j_1^{1/2}h_{j_1-1,j_2,j_3}(\mathbf{p})\\&\quad+2^{-1/2}\left[\frac{p_1}{2}(\mathbf{p}^2+m^2)^{-1}(\mathbf{p}^2+m^2)^{1/4}h_{j_1,j_2,j_3}(\mathbf{p})\right]\\&=j_1^{1/2}g_{j_1-1,j_2,j_3}(p)+\frac{2^{-1/2}}{2}p_1(\mathbf{p}^2+m^2)^{-1}g_{j_1,j_2,j_3}(p)\end{aligned}$$

Similarly,

$$2^{-1/2}\left(p_1-\frac{\partial}{\partial p_1}\right)h_{j_1,j_2,j_3}(\mathbf{p})=(j_1+1)^{1/2}h_{j_1+1,j_2,j_3}$$

and the proof is analogous. ■

The standard basis $e_1, e_2, e_3$ for $V$ corresponds to the color states since they are the eigenvectors of the color observable. We define the *three-dimensional Fourier transform* $F_3: V\to V$ as the matrix

$$F_3=\frac{1}{\sqrt{3}}\begin{bmatrix}b & \bar{b} & 1\\ \bar{b} & b & 1\\ 1 & 1 & 1\end{bmatrix},\qquad b=e^{2\pi i/3}$$

We call the matrix $P_c=F_3^*CF_3$ the *color momentum observable* [1, 2]. The eigenvalues of $P_c$ are $c_1, c_2, c_3$ with corresponding eigenvectors $f_j=F_3^*e_j$, $j=1,2,3$. These give the color momentum states and have the components $f_1=3^{-1/2}(\bar{b},b,1)$, $f_2=3^{-1/2}(b,\bar{b},1)$, $f_3=3^{-1/2}(1,1,1)$. We define $A(f_j)=\Psi(j)$ and $A^*(f_j)=\Psi^*(j)$, $j=1,2,3$. For arbitrary $f\in V$, we define

$$A(f)=\sum\langle f_j,f\rangle A(f_j)$$
$$A^*(f)=\sum\langle f,f_j\rangle A^*(f_j)$$

It is easy to check that $[A(f),A^*(g)]=\langle g,f\rangle$. We also define the *field operators*

$$\pi(f)=2^{-1/2}[A(f)+A^*(f)]$$
$$\psi(f)=-2^{-1/2}i[A(f)-A^*(f)]$$

In the next theorem, for $f\in V$, we write $\mathbf{f}=(\langle f,f_1\rangle,\langle f,f_2\rangle,\langle f,f_3\rangle)$ and $\bar{\mathbf{f}}=(\langle f_1,f\rangle,\langle f_2,f\rangle,\langle f_3,f\rangle)$.

*Theorem 7.* For $f \in V$ we have

$$A(f) = \frac{1}{\sqrt{2}}[\bar{\mathbf{f}} \cdot \mathbf{p} - \tfrac{1}{2}(\mathbf{p}^2 + m^2)^{-1}\bar{\mathbf{f}} \cdot p + \bar{\mathbf{f}} \cdot \nabla]$$

$$A^*(f) = \frac{1}{\sqrt{2}}[\mathbf{f} \cdot \mathbf{p} + \tfrac{1}{2}(\mathbf{p}^2 + m^2)^{-1}\mathbf{f} \cdot p - \mathbf{f} \cdot \nabla]$$

$$\pi(f) = \operatorname{Re} \mathbf{f} \cdot \mathbf{p} + \frac{i}{2}(\mathbf{p}^2 + m^2)^{-1} \operatorname{Im} \mathbf{f} \cdot \mathbf{p} - i \operatorname{Im} \mathbf{f} \cdot \nabla$$

$$\psi(f) = -\operatorname{Im} \mathbf{f} \cdot \mathbf{p} + \frac{i}{2}(\mathbf{p}^2 + m^2)^{-1} \operatorname{Re} \mathbf{f} \cdot \mathbf{p} - i \operatorname{Re} \mathbf{f} \cdot \nabla$$

*Proof.* These follow from Lemma 6 and the definitions. ■

From Theorem 7, if $f \in V$ is a real vector, we have

$$\pi(f) = \mathbf{f} \cdot \mathbf{p}, \qquad \psi(f) = \frac{i}{2}(\mathbf{p}^2 + m^2)^{-1}\mathbf{f} \cdot \mathbf{p} - i\mathbf{f} \cdot \nabla$$

In particular,

$$\pi(f_j) = p_j, \qquad \psi(f_j) = \frac{i}{2}(\mathbf{p}^2 + m^2)^{-1} p_j - i\frac{\partial}{\partial p_j}, \qquad j = 1, 2, 3$$

The *jth particle number observable* becomes

$$N(f_j) = A^*(f_j)A(f_j)$$
$$= \frac{1}{2}\left[p_j^2 - \frac{\partial^2}{\partial p_j^2} - I - \frac{5p_j^2}{4(\mathbf{p}^2 + m^2)^2} + \frac{1}{2(\mathbf{p}^2 + m^2)} + \frac{p_j}{(\mathbf{p}^2 + m^2)}\frac{\partial}{\partial p_j}\right]$$

and the *total number observable* is

$$N = \sum N(f_j) = \frac{1}{2}\left[\mathbf{p}^2 - \nabla^2 - 3I - \frac{5\mathbf{p}^2}{4(\mathbf{p}^2 + m^2)^2} + \frac{3}{2(\mathbf{p}^2 + m^2)} + \frac{1}{\mathbf{p}^2 + m^2}\mathbf{p} \cdot \nabla\right]$$

We now define the second quantization of linear operators on $V$. If $B: V \to V$ is a linear operator, we define

$$\Gamma(B) = \sum A^*(f_k)A(Bf_k)$$

For example, $\Gamma(I) = N$, $\Gamma(P_j) = N(f_j)$, $j = 1, 2, 3$, where $P_j$ is the projection onto $f_j$.

*Lemma 8.* (a) The definition of $\Gamma(B)$ is independent of the basis.

(b) If $B$ is self-adjoint with eigenpairs $\lambda_j$, $g_j$, $j = 1, 2, 3$, then

$$\Gamma(B) = \sum_{j,k,m} \lambda_k \langle f_m, g_k \rangle \langle g_k, f_j \rangle A^*(f_j)A(f_m)$$
$$= \sum_k \lambda_k A^*(g_k)A(g_k)$$

(c) If $B$ is self-adjoint, then so is $\Gamma(B)$.

*Proof.* (a) Let $g_j$, $j=1,2,3$, be another orthonormal basis. Then

$$\begin{aligned}\sum A^*(g_k)A(Bg_k) &= \sum_k A^*\left(\sum_j \langle g_k, f_j\rangle f_j\right) A\left(B\sum_m \langle g_k, f_m\rangle f_m\right)\\ &= \sum_{j,k,m} \langle g_k, f_j\rangle\langle f_m, g_k\rangle A^*(f_j)A(Bf_m)\\ &= \sum_{j,m} A^*(f_j)A(Bf_m)\sum_k \langle f_m, g_k\rangle\langle g_k, f_j\rangle\\ &= \sum_j A^*(f_j)A(Bf_j)\end{aligned}$$

(b) By part (a) we have

$$\begin{aligned}\Gamma(B) &= \sum A^*(g_k)A(Bg_k) = \sum \lambda_k A^*(g_k)A(g_k)\\ &= \sum_k \lambda_k A^*\left(\sum_j \langle g_k, f_j\rangle f_j\right) A\left(\sum_m \langle g_k, f_m\rangle f_m\right)\\ &= \sum_{j,k,m} \lambda_k \langle f_m, g_k\rangle\langle g_k, f_j\rangle A^*(f_j)A(f_m)\end{aligned}$$

(c) This follows from part (b). ■

Let $B\colon V\to V$ be self-adjoint with matrix $B_{jk}$; that is, $Bf_k=\sum_j B_{jk}f_j$. On the six-dimensional space of operators spanned by $\Psi(j)$, $\Psi^*(j)$, $j=1,2,3$, define a linear operator $\hat{B}$ by $(\hat{B}\Psi)(k)=A(Bf_k)$, $(\hat{B}\Psi^*)(k)=A^*(Bf_k)$.

*Lemma 9.*

(a) $$(\hat{B}\Psi)(k)=\sum_j B_{jk}\Psi(j),\ (\hat{B}\Psi^*)(k)=\sum_j \bar{B}_{kj}\Psi^*(j)$$

(b) $$\Gamma(B)=\sum \Psi^*(k)(\hat{B}\Psi)(k)=\sum (\hat{B}\Psi^*)(k)\Psi(k)=\sum_{j,k} B_{kj}\Psi^*(k)\Psi(j)$$

*Proof.*

(a) $$(\hat{B}\Psi)(k)=A(Bf_k)=A\left(\sum_j B_{jk}f_j\right)=\sum_j \bar{B}_{jk}A(f_j)=\sum_j B_{kj}\Psi(j)$$

$$(\hat{B}\Psi^*)(k)=A^*(Bf_k)=A^*\left(\sum_j B_{jk}f_j\right)=\sum_j B_{jk}A^*(f_j)=\sum_j \bar{B}_{kj}\Psi^*(j)$$

(b) $$\begin{aligned}\Gamma(B) &= \sum A^*(f_k)A(Bf_k)=\sum \Psi^*(k)(\hat{B}\Psi)(k)\\ &= \sum_k \Psi^*(k)\sum_j B_{kj}\Psi(j)\\ &= \sum_{j,k} B_{kj}\Psi^*(k)\Psi(j)=\sum_j\sum_k \bar{B}_{jk}\Psi^*(k)\Psi(j)\\ &= \sum (\hat{B}\Psi^*)(j)\Psi(j)\end{aligned}$$ ■

A function $v(j, k) = v(k, j)$, $j, k = 1, 2, 3$, corresponds to a two-particle potential on $V \otimes V$. We define the second quantization of $v$ by

$$\Gamma(v) = \tfrac{1}{2} \sum_{j,k} v(j, k) \Psi^*(k) \Psi^*(j) \Psi(j) \Psi(k)$$

We now consider dynamics. In the Heisenberg picture, we define

$$\Psi(k, t) = e^{i\Gamma(H)t} \Psi(k)\, e^{-i\Gamma(H)t}$$

where $H$: $V \to V$ is a self-adjoint matrix corresponding to the Hamiltonian. Notice that

$$[\Psi(k, j), \Psi(j, t)] = [\Psi^*(k, t), \Psi^*(j, t)] = 0$$

$$[\Psi(k, t), \Psi^*(j, t)] = \delta_{kj}$$

Differentiation gives

$$i \frac{\partial \Psi(k, t)}{\partial t} = [\Psi(k, t), \Gamma(H)]$$

For $B$: $V \to V$ we extend the definition of $\hat{B}$ by

$$(\hat{B}\Psi)(k, t) = e^{i\Gamma(H)t} (\hat{B}\Psi)(k)\, e^{-i\Gamma(H)t} = \sum_j B_{kj} \Psi(j, t)$$

Suppose $H_0$: $V \to V$ is a free-particle Hamiltonian and $v(j, k)$ corresponds to a two-particle potential. We then take $\tilde{H} = \Gamma(H_0) + \Gamma(v)$ as the second quantized Hamiltonian. The next theorem shows that rigorous results can be obtained in the finite-dimensional theory where the corresponding results in the usual infinite-dimensional theory are only heuristic.

*Theorem 10.* If $\Psi(k, t) = e^{i\tilde{H}t} \Psi(k)\, e^{-i\tilde{H}t}$, then

$$i \frac{\partial \Psi(k, t)}{\partial t} = (\hat{H}_0 \Psi)(k, t) + \left[ \sum_j v(j, k) \Psi^*(j, t) \Psi(j, t) \right] \Psi(k, t)$$

*Proof.* Applying Lemma 9(b) we have

$$\tilde{H} = \sum_{j,k} H_{0kj} \Psi^*(k) \Psi(j) + \tfrac{1}{2} \sum_{j,k} v(j, k) (\Psi^*(k) \Psi^*(j) \Psi(k)$$

Since $\tilde{H}$ commutes with $e^{i\tilde{H}t}$ we may write

$$\begin{aligned} \tilde{H} &= e^{i\tilde{H}t} \tilde{H} e^{-i\tilde{H}t} \\ &= \sum_{j,k} H_{0kj} \Psi^*(k, t) \Psi(j, t) \\ &\qquad + \tfrac{1}{2} \sum_{j,k} v(j, k) \Psi^*(k, t) \Psi^*(j, t) \Psi(j, t) \Psi(k, t) \end{aligned}$$

Hence

$$\begin{aligned}
i\frac{\partial\Psi(n,t)}{\partial t} &= \Psi(n,t)\tilde{H}-\tilde{H}\Psi(n,t)\\
&=\sum_{j,k} H_{0kj}[\Psi(n,t)\Psi^*(k,t)\Psi(j,t)-\Psi^*(k,t)\Psi(j,t)\Psi(n,t)]\\
&\quad+\tfrac{1}{2}\sum_{j,k} v(j,k)[\Psi(n,t)\Psi^*(k,t)\Psi^*(j,t)\Psi(j,t)\Psi(k,t)\\
&\quad-\Psi^*(k,t)\Psi^*(j,t)\Psi(j,t)\Psi(k,t)\Psi(n,t)]\\
&=\sum_{j,k} H_{0kj}[\Psi(n,t)\Psi^*(k,t)]\Psi(j,t)\\
&\quad+\tfrac{1}{2}\sum_{j,k} v(j,k)[\Psi(n,t),\Psi^*(k,t)\Psi^*(j,t)]\Psi(j,t)\Psi(k,t)
\end{aligned}$$

Now

$$\begin{aligned}
[\Psi(n,t),\Psi^*(k,t)\Psi^*(j,t)] &= [\Psi(n,t),\Psi^*(k,t)]\Psi^*(j,t)\\
&\quad+\Psi^*(k,t)[\Psi(n,t),\Psi^*(j,t)]\\
&=\Psi^*(j,t)\delta_{n,k}+\Psi^*(k,t)\delta_{n,j}
\end{aligned}$$

Thus

$$\begin{aligned}
i\frac{\partial\Psi(n,t)}{\partial t} &= \sum_j H_{0nj}\Psi(j,t)+\tfrac{1}{2}\sum_j v(j,n)\Psi^*(j,t)\Psi(j,t)\Psi(n,t)\\
&\quad+\tfrac{1}{2}\sum_k v(n,k)\Psi^*(k,t)\Psi(m,t)\Psi(k,t)\\
&=(\hat{H}_0\Psi)(n)+\left(\sum_j v(j,n)\Psi^*(j,t)\Psi(j,t)\right)\Psi(n,t) \quad \blacksquare
\end{aligned}$$

## 5. FINITE-DIMENSIONAL DIRAC THEORY

On $\mathbb{C}^3$ we have defined the color observable $C=\mathrm{diag}(c_1, c_2, c_3)$ and the color momentum observable $P_C=F_3^*CF_3$ with eigenvectors $e_1, e_2, e_3$ and $f_1, f_2, f_3$, respectively. We define the *Klein-Gordon Hamiltonian* on $\mathbb{C}^3$ by $H_{KG}=[P_C^2+m^2I]^{1/2}$. The *second quantized particle Klein-Gordon Hamiltonian* is defined as $\Gamma(H_{KG})$ and the *second quantized wave Klein-Gordon Hamiltonian* is defined as

$$W(H_{KG})=[\pi(f_1)^2+\pi(f_2)^2+\pi(f_3)^2+m^2]^{1/2}=[\mathbf{p}^2+m^2]^{1/2}$$

In order to motivate a finite-dimensional Dirac theory, we consider a quark model discussed in [1, 2]. As shown in [1, 2], quark states are given by vectors in $\mathbb{C}^{72}$. The states are conveniently described by four parameters: color $(r, y, b)$, flavor $(d, u, s, c, b, t)$, spin (up, down), type (particle, antiparticle). In this way we write $\mathbb{C}^{72}$ as the tensor product of four subspaces, $\mathbb{C}^{72} = \mathbb{C}^3 \otimes \mathbb{C}^6 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, the *color, flavor, spin,* and *type subspaces.* On the spin subspace $\mathbb{C}^2$, there are three *spin observables,* which are given by $\tau_1$, $\tau_2$, $\tau_3$ of Section 2. On the type subspace $\mathbb{C}^2$, particles are represented by the vector $\psi_1 = (1, 0)$ and antiparticles by the vector $\psi_2 = (0, 1)$. The *type observable* $T$ has eigenvalue 1 for particles and eigenvalue $-1$ for antiparticles. Thus

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The *conjugation operator* $K$ takes particles to antiparticles and antiparticles to particles; that is, $K\psi_1 = \psi_2$ and $K\psi_2 = \psi_1$. Hence,

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

On the color subspace $\mathbb{C}^3$, the color momentum observable has the spectral representation $P_C = c_1 P_1 + c_2 P_2 + c_3 P_3$, where $P_1, P_2, P_3$ are the one-dimensional projections onto $f_1, f_2, f_3$, respectively. We shall not need the flavor subspace $\mathbb{C}^6$ in our present discussion except to comment that the constant $m$ which follows has one of the six possible flavor values corresponding to the six flavors [2].

We now define the *Dirac Hamiltonian* as

$$H_D = c_1 P_1 \otimes \tau_1 \otimes K + c_2 P_2 \otimes \tau_2 \otimes K + c_3 P_3 \otimes \tau_3 \otimes K + mI \otimes I \otimes T$$

This expression can be simplified by defining the following $4\times 4$ matrices:

$$\alpha_0 = I \otimes T = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \qquad \alpha_j = \tau_j \otimes K = \begin{bmatrix} 0 & \tau_j \\ \tau_j & 0 \end{bmatrix}, \qquad j = 1, 2, 3$$

We then have

$$H_D = c_1 P_1 \otimes \alpha_1 + c_2 P_2 \otimes \alpha_2 + c_3 P_3 \otimes \alpha_3 + mI \otimes \alpha_0$$

Of course, the $\alpha_j$, $j = 0, 1, 2, 3$ are the Dirac matrices and satisfy the relations $[\alpha_j, \alpha_k]_+ = 0$, $\alpha_j^2 = I$, $j = 0, 1, 2, 3$. Because of these relations, we have

$$\begin{aligned} H_D^2 &= c_1^2 P_1 \otimes I + c_2^2 P_2 \otimes I + c_3^2 P_3 \otimes I + m^2 I \otimes I \\ &= (P_C^2 + m^2 I) \otimes I = H_{KG}^2 \otimes I \end{aligned}$$

We define

$$\begin{aligned} \Gamma(H_D) &= c_1 \Gamma(P_1) \otimes \alpha_1 + c_2 \Gamma(P_2) \otimes \alpha_2 + c_3 \Gamma(P_3) \otimes \alpha_3 + m\Gamma(I) \otimes \alpha_0 \\ &= c_1 N(f_1) \otimes \alpha_1 + c_2 N(f_2) \otimes \alpha_2 + c_3 N(f_3) \otimes \alpha_3 + mN \otimes \alpha_0 \end{aligned}$$

and

$$
\begin{aligned}
W(H_D) &= \pi(f_1)\otimes\alpha_1+\pi(f_2)\otimes\alpha_2+\pi(f_3)\otimes m\otimes\alpha_0\\
&= p_1\otimes\alpha_1+p_2\otimes\alpha_2+p_3\otimes\alpha_3+m\otimes\alpha_0
\end{aligned}
$$

The proof of the next theorem is tedious but straightforward.

*Theorem 11.* The eigenvalues of $H_D$ are double eigenvalues and consist of the numbers $\lambda_{j\pm}=\pm 2^{1/2}(c_j+m)$, $j=1,2,3$. The normalized eigenvectors corresponding to $\lambda_{j\pm}$ are $f_j\otimes g_{jk\pm}$, $j=1,2,3$; $k=1,2$, where

$$
\begin{aligned}
g_{11+} &= (4-2^{3/2})^{-1/2}(1,0,0,2^{1/2}-1)\\
g_{12+} &= (4-2^{3/2})^{-1/2}(0,1,2^{1/2}-1,0)\\
g_{11-} &= (4+2^{3/2})^{-1/2}(1,0,-2^{1/2}-1,0)\\
g_{12-} &= (4+2^{3/2})^{-1/2}(0,1,-2^{1/2}-1,0)\\
g_{21+} &= (4-2^{3/2})^{-1/2}(1,0,0,i(2^{1/2}-1))\\
g_{22+} &= (4-2^{3/2})^{-1/2}(0,1,i(2^{1/2}-1),0)\\
g_{21-} &= (4+2^{3/2})^{-1/2}(1,0,0,-i(2^{1/2}+1))\\
g_{22-} &= (4+2^{3/2})^{-1/2}(0,1,i(2^{1/2}+1),0)\\
g_{31+} &= (4-2^{3/2})^{-1/2}(1,0,2^{1/2}-1,0)\\
g_{32+} &= (4-2^{3/2})^{-1/2}(0,1,0,1-2^{1/2})\\
g_{31-} &= (4+2^{3/2})^{-1/2}(1,0,-2^{1/2}-1,0)\\
g_{32-} &= (4+2^{3/2})^{-1/2}(0,1,0,2^{1/2}+1,0)
\end{aligned}
$$

One also has the analogs of other Hamiltonians in common use. For example, the Foldy-Wouthuysen form of the Dirac Hamiltonian is

$$H_{FW}=[P_C^2+m^2I]^{1/2}\otimes\alpha_0$$

*Theorem 12.* The eigenvalues of $H_{FW}$ are double eigenvalues and consist of the numbers $\lambda_{j\pm}=c_j+m$, $j=1,2,3$. The normalized eigenvectors corresponding to $\lambda_{j\pm}$ are $f_j\otimes h_{jk\pm}$, $j=1,2,3$; $k=1,2$, where

$$
\begin{aligned}
h_{j1+} &= (1,0,0,0)\\
h_{j2+} &= (0,1,0,0)\\
h_{j1-} &= (0,0,1,0)\\
h_{j2-} &= (0,0,0,1) \qquad j=1,2,3
\end{aligned}
$$

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